OTS: 60-11,802

JPRS: 2897

20 June 1960

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- USSR -

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Distributed by:

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WASHINGTON 25, D. C.

Price: $0.50

U. S. JOINT PUBLICATIONS RESEARCH SERVICE
205 EAST 42nd STREET, SUITE 300
NEW YORK 17, N. Y.

JPRS: 2897

CSO: 4006-N/b

AXIALLY SYMMETRICAL MERIDIAN FLOW OF A CONDUCTING FLUID. BALANCING THE PARAMETERS OF THE ROTATIONAL FLOW OF A VISCOUS FLUID.

[This is a translation of the article written by G. L. Grodizovskiy, A. N. Dyukalov, V. V. Tokarev, and A. I. Tolstykh in the Bulletins of the Academy of Sciences USSR, Division of Technical Sciences, Mechanics and Mechanical Engineering, Moscow, No. 1, 1960, pages 41-46.]

1. Electrodynamic equations of magnetic hydrodynamics have the form [1]

$$\operatorname{div}\mathbf{H} = 0, \qquad \frac{\partial\mathbf{H}}{\partial t} = \operatorname{rot}[\mathbf{v}\times\mathbf{H}] + \frac{c^2}{4\pi\sigma}\Delta\mathbf{H} \tag{1.1}$$

where H is magnetic field vector, v - velocity vector of the conducting fluid, σ - conductivity, c - velocity of light.

Electric current density

$$\mathbf{j} = \frac{c}{4\pi}\operatorname{rot}\mathbf{H} \tag{1.2}$$

Let us consider steady, axially symmetrical, meridian flows in which the tangential components of the velocity and currents is absent

$$v_\vartheta = 0, \qquad j_\vartheta = 0 \tag{1.3}$$

Making use of condition (1.3), we obtain from equation (1.1) the following relations:

$$v_x H_r - v_r H_x = 0 \tag{1.4}$$

$$\frac{\partial(v_x H_\vartheta)}{\partial x} + \frac{\partial(v_r H_\vartheta)}{\partial r} = \frac{c^2}{4\pi\sigma}\left(\frac{\partial^2 H_\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial H_\vartheta}{\partial r} - \frac{H_\vartheta}{r^2} + \frac{\partial^2 H_\vartheta}{\partial x^2}\right) \tag{1.5}$$

In the case of the meridian flow of an incompressible fluid at the velocity $v_x = v_0$ (for example, between two coaxial cylinders with radii a and b) we arrive at the system of equations

$$H_r = 0, \qquad H_x = 0 \tag{1.6}$$

$$\frac{\partial H_\vartheta}{\partial x} - \frac{\delta}{2}\frac{\partial^2 H_\vartheta}{\partial x^2} = \frac{\delta}{2}\left(\frac{\partial^2 H_\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial H_\vartheta}{\partial r} - \frac{H_\vartheta}{r^2}\right) \quad \left(\delta = \frac{c^2}{2\pi\sigma v_0}\right) \tag{1.7}$$

and the equations of motion will be written in the form

$$\frac{\partial p}{\partial r} = -\frac{1}{4\pi}\left(H_\vartheta \frac{\partial H_\vartheta}{\partial r} + \frac{H_\vartheta^2}{r}\right), \qquad \frac{\partial p}{\partial x} = -\frac{1}{4\pi} H_\vartheta \frac{\partial H_\vartheta}{\partial x} \tag{1.8}$$

In the plane case, equations (1.8) are simultaneous when H is arbitrary. In the axially symmetrical case this is achieved approximately when the values of H_ϑ^2/r are small.

We seek the solution of equation (1.7) by Fourier method in the form

$$H_\vartheta(x, r) = X(x) R(r) \tag{1.8a}$$

Computations give $X(x) = Ae^{k_1 x} + Be^{k_2 x}$

$$R(r) = CJ_1(\lambda r) + DY_1(\lambda r) \qquad \left(k_{1,2} = \frac{1 \mp \sqrt{1 + \delta^2\lambda^2}}{\delta}, \ \delta = \frac{c^2}{2\pi\sigma v_0}\right) \tag{1.9}$$

where J_1, Y_1 are Bessel functions of the first order, of the first and second kind.

In addition, the linear function of the radius $H_\vartheta = Kr$ satisfies equation (1.8). Therefore, the solution may be written in the form

$$H_\vartheta(x, r) = \sum_{i=1}^{\infty} (A_i e^{k_{1,i}x} + B_i e^{k_{2,i}x})[C_i J_1(\lambda_i r) + D_i Y_1(\lambda_i r)] + Kr \tag{1.10}$$

The values of λ_i are determined from the boundary conditions of the problem. The cylinder surfaces limiting the flow are considered to be non-conducting. Therefore, the density on them of the radial component of the current, j_r, must equal zero

$$j_r(x, a) = 0, \qquad j_r(x, b) = 0 \tag{1.11}$$

or, in accordance with (1.2), we will write the boundary conditions in the form

$$\left(\frac{\partial H_\vartheta}{\partial x}\right)_{r=a} = \left(\frac{\partial H_\vartheta}{\partial x}\right)_{r=b} = 0$$

Hence

$$J_1(\lambda_i a) Y_1(\lambda_i b) - J_1(\lambda_i b) Y_1(\lambda_i a) = 0 \tag{1.12}$$

Coefficient K is determined by the total axial current

$$I = K\pi (b^2 - a^2) \tag{1.12a}$$

Accordingly, solution (1.10) is written in the form

$$H_\vartheta(x,\ r) = \sum_{i=1}^{\infty} (A_i' e^{k_{1,i}x} + B_i' e^{k_{2,i}x}) [Y_1(\lambda_i a) J_1(\lambda_i r)] -$$
$$- J_1(\lambda_i a) Y_1(\lambda_i r) + \frac{I}{\pi (b^2 - a^2)} r \tag{1.13}$$

where A_i' and B_i' are new constants which are determined, for example, by the given distribution of current in two cross sections.

Let us note the effect of the flow velocity v_0 on the flow of electric current in the conducting fluid. Let the electric current flow off with a nonuniform density distribution over the radius in cross section $x_1 = 0$ which has some first system of electrodes.

The second electrode is located on $x_2' = +\infty$ or on $x_2'' = -\infty$ which corresponds to the electric current flowing off with the flow or against the flow. Since $k_1 < 0$ and $k_2 > 0$, we have from the boundary condition of the flow parameters at infinity:

In the former case

$$H_\vartheta' \sim \exp\left[(1 + \sqrt{(1 + \delta^2\lambda^2)}) \frac{x}{\delta}\right] \tag{1.14}$$

In the latter case

$$H_\vartheta'' \sim \exp\left[(1 + \sqrt{1 + \delta^2\lambda^2}) \frac{x}{\delta}\right] \tag{1.15}$$

i.e. the non-uniformity of the current density is attenuated more rapidly against the flow than with the flow of the fluid.

With the decrease of the relation $(b - a)/r$ in the limit we go over to the plane case for which the solution is exact.

2. Let us show similarly to the well known exact solutions for the flows of viscous incompressible fluid [2-4] that in the case of the meridian flows of incompressible conducting fluid, the equations of magnetohydrodnamics

$$v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{4\pi\rho} H_\vartheta \frac{\partial H_\vartheta}{\partial x} \tag{2.1}$$

$$v_x \frac{\partial v_r}{\partial x} + v_r \frac{\partial v_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{1}{4\pi\rho} H_\vartheta \frac{1}{r}\frac{\partial (rH_\vartheta)}{\partial r} \tag{2.2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0 \tag{2.3}$$

$$\frac{\partial}{\partial x}(v_x H_\vartheta) + \frac{\partial}{\partial r}(v_r H_\vartheta) = \frac{c^2}{4\pi\sigma}\left(\frac{\partial^2 H_\vartheta}{\partial r^2} + \frac{\partial^2 H_\vartheta}{\partial x^2} + \frac{1}{r}\frac{\partial H_\vartheta}{\partial r} - \frac{H_\vartheta}{r^2}\right) \tag{2.4}$$

permit the following class of automatically scaled solutions: independent variables and the defining parameters are r, x, c^2/σ, ρ.

Let us consider in a similar manner [2] motions which are defined by parameters r, x, c^2/σ, ρ and, in addition, by only one

dimensional constant A

$$[A] = L^{q_1} T^{q_2} M \tag{2.5}$$

When

$$2q_2 + q_1 + 3 = 0 \tag{2.6}$$

there exists only one dimensionless variable $\zeta = x/r$.

[2] Therefore, there exists on the basis of the dimension theory the following class of automatically scaled solutions[1)]

$$v_x = \frac{c^2}{4\pi\sigma}\frac{1}{r}f(\zeta), \qquad v_r = \frac{c^2}{4\pi\sigma}\frac{1}{r}\varphi(\zeta) \tag{2.7}$$

$$H_\theta = \frac{c^2}{4\pi\sigma}\frac{\sqrt{2\pi\rho}}{r}\Phi(\zeta), \qquad p = \left(\frac{c^2}{4\pi\sigma}\right)^2\frac{\rho}{r^2}P(\zeta) \tag{2.8}$$

where functions f, φ, Φ, P are defined by the following system of ordinary differential equations:

$$\begin{gathered} f'(f-\zeta\varphi) - f\varphi + P' + \tfrac{1}{2}\Phi\Phi' = 0 \\ \varphi'(f-\zeta\varphi) - \varphi\varphi - \zeta P' - 2P - \tfrac{1}{2}\Phi\Phi' = 0 \\ f' - \zeta\varphi' = 0 \\ (1+\zeta^2)\Phi'' + (3\zeta - f + \zeta\varphi)\Phi' + 2\varphi\Phi = 0 \end{gathered} \tag{2.9}$$

Let us introduce function $\chi = f - \zeta\varphi$ which is associated with the function of the current ψ by the relation:

$$\psi = r\chi \qquad (f = \chi - \zeta\chi', \ \varphi = -\chi') \tag{2.10}$$

The first one of the equations (2.9) may be integrated by making use of the third one of them. We will obtain

$$\chi(\chi - \zeta\chi') + P + \tfrac{1}{4}\Phi^2 = \gamma = \text{const} \tag{2.11}$$

and will represent the remaining equations (2.9) in the form

$$(1+\zeta^2)(\chi^2)'' + \zeta(\chi^2)' - 4\chi^2 - \Phi^2 = -4\gamma \tag{2.12}$$

$$(1+\zeta^2)\Phi'' + (3\zeta - \chi)\Phi' - 2\chi'\Phi = 0 \tag{2.13}$$

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1) It is interesting to note that a broader class of automatically scaled solutions with respect to parameter $\zeta_1 = x^n/r$ corresponds to the approximate statement of the problem [5] when $R = 4\pi\sigma I/c^3\sqrt{\rho} \gg 1$.

In the class of the flows being considered, the equipotential surfaces are defined with the aid of Ohm's generalized law:

$$j_r = -\frac{c}{4\pi}\frac{\partial H_\vartheta}{\partial x} = \sigma\left(E_r - \frac{1}{c} v_x H_\vartheta\right) \tag{2.14}$$

$$j_x = \frac{c}{4\pi}\frac{1}{r}\frac{\partial (rH_\vartheta)}{\partial r} = \sigma\left[E_x + \frac{1}{c} v_r H_\vartheta\right] \tag{2.15}$$

Whence

$$U = \frac{1}{r}[(1+\zeta^2)\Phi' - \chi\Phi] \tag{2.16}$$

where U is the potential of the electric field E.

The direction of the current along the rays passing through the origin of coordinates is characteristic of the flows being considered. Let us examine some examples.

Conical Discharge in Unbounded Medium. - Let the electrical discharge be concentrated in the conical region (Figure 1) having the central angle of $\varphi_0 = \operatorname{arc} \operatorname{Tg} \zeta_0$. Outside the discharge region the magnetic field coincides with the field of the straight current

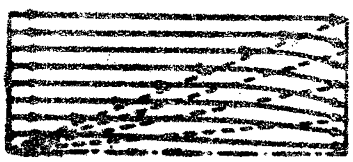


Figure 1.

$$H_\vartheta = \frac{2I}{cr} = \frac{A}{r},$$

$$\Phi(\zeta) = \frac{4\sqrt{2\pi\varepsilon}}{c^3\sqrt{\rho}} I = KI, \qquad \zeta<\zeta_0 \tag{2.17}$$

Then from equations (2.12) and (2.13) follows $\chi(\zeta) = f_0$ when $\zeta < \zeta_0$ and the problem is reduced to the integration of equations (2.12) and (2.13) within the conical region $\zeta > \zeta_0$ with the boundary conditions:

$$\Phi(\zeta_0) = KI, \qquad \chi(\zeta_0) = f_0$$

(condition of the continuity of the solution)

$$\chi'(\zeta_0) = 0 \tag{2.18}$$

$\lim \Phi(\zeta)\zeta = 0, \quad \lim \chi'(\zeta)\zeta = 0$ when $\zeta \to \infty$ (condition on the axis)

The relations cited completely define the solution and determine the connection between f_0 and I.

When the central angles of the conical region are small ($\zeta_0 \gg 1$), we may write the following solution confining ourselves to the first terms of the decomposition with respect to ζ^{-1}

$$\chi(\zeta) = \alpha + (f_0 - \alpha)\left(\frac{\zeta_0}{\zeta}\right)^2\left[2 - \left(\frac{\zeta_0}{\zeta}\right)^2\right] + 0(\zeta^{-5})$$

$$\Phi(\zeta) = \frac{\beta}{\zeta^2}\left[1 - \frac{2}{3}\alpha\frac{1}{\zeta} + \frac{1}{4}(\alpha^2 - 3)\frac{1}{\zeta^2}\right] + 0(\zeta^{-5}) \tag{2.19}$$

Here

$$\alpha^2 = f_0^2 + \frac{1}{4} K^2 I^2, \qquad \beta^2 = 24\zeta_0^2 (f_0 - \alpha) [(2f_0 - 3\alpha)\zeta_0^2 + \alpha]$$

and f_o is defined by the relation

$$\beta \left[\zeta_0^2 - \frac{2}{3} \alpha \zeta_0 + \frac{1}{4} (\alpha^2 - 3) \right] = K I \zeta_0^4 \tag{2.20}$$

The form of the lines of the current in such a flow is shown in Figure 1. Conical discharge produces the motion of the fluid (ejection). The necessity for the occurrence of the ejection has been demonstrated in the work [5].

Discharge in a Conical Channel with Non-Conducting Walls. - In this case the boundary conditions have the form

$$\chi(\zeta_0) = 0, \qquad \Phi(\zeta_0) = KI$$
$$\lim \zeta \Phi(\zeta) = 0, \qquad \lim \zeta \chi'(\zeta) = 0 \quad \text{при} \quad \zeta \to \infty \tag{2.21}$$

and the constant γ will be determined by the "power" of the impulse source at the origin of the coordinates (see similarly in [3]). When $\zeta_o \gg 1$, one may write the following approximate solution:

$$\chi(\zeta) = a + c \frac{1}{\zeta^2} - \zeta_0^2 (c + a\zeta_0^2) \frac{1}{\zeta^4} + 0(\zeta^{-6})$$
$$\Phi(\zeta) = \frac{b}{\zeta^2} \left[1 - \frac{2}{3} a \frac{1}{\zeta} + \frac{1}{4} (a^2 - 3) \frac{1}{\zeta^2} \right] + 0(\zeta^{-5}) \tag{2.22}$$

where

$$a = \sqrt{\frac{1}{4} K^2 I^2 + P(\zeta_0)}, \qquad b = \frac{12\zeta_0^2 KI}{a(3a - 8\zeta_0) + 12\zeta_0^2 - 9}$$
$$c = \frac{1}{2} \left[a(2\zeta_0^2 - 1) - \sqrt{4a^2\zeta_0^2 (3\zeta_0^2 - 1) + a^2 + \frac{1}{8} b^2} \right] \tag{2.23}$$

A picture of the current lines in the case of an impulse source of zero power is given in Figure 2. The flow analyzed is characterized by the accumulation of the flow toward the axis. An approximate solution can be written similarly when $\zeta_o \ll 1$. A picture of the line of the current is given in Figure 3:

$$\chi(\zeta) = a\left(\zeta - \frac{1}{\zeta_0} \zeta^2\right) + 0(\zeta^3), \qquad \Phi(\zeta) = b - \frac{6a^2}{b\zeta_0} \zeta + ba\zeta^2 + 0(\zeta^3)] \tag{2.24}$$

where

$$b = KI$$

$$a^2 = \frac{1}{72}\left[12b^2 + b^4\zeta_0^4 + b^3\zeta_0^2\sqrt{24 + b^2\zeta_0^4}\right] \approx \frac{1}{72}[12b^2 + \sqrt{24}b^3\zeta_0^2] \quad (2.25)$$

3. The analogy of the electrodynamic and viscous flows of a fluid is known. In the case of a viscous fluid, the balancing of the parameters of a rotational flow corresponds to equation (1.7).

Let us consider, for example, the axially symmetrical flow of a viscous fluid which moves inside a cylindrical boundary having radius r_0 at the constant longitudinal velocity $v_x = v_0$ without friction against the walls. (The simplifying assumption concerning the absence of friction against the walls is achieved in some cases in practice. For example, freely rotating walls are used in centrifugal pumps). From the condition of continuity

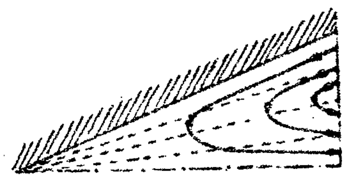


Figure 2.

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_\vartheta}{\partial \vartheta} + \frac{\partial v_x}{\partial x} = 0 \quad (3.1)$$

Making use of the conditions

$$\partial v_x / \partial x = 0 \quad (v_x = v_0), \qquad \partial v_\vartheta / \partial \vartheta = 0 \quad \text{axial symmetry} \quad (3.1a)$$

and of the condition on the wall $(v_r = 0)_{r=r_0}$, we find that the radial velocity equals zero everywhere. In the absence of mass forces, the equation of motion will be written similarly to (1.7) in the form

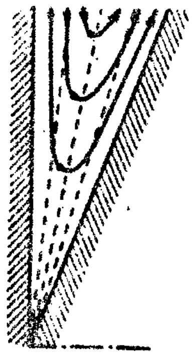
$$v_0 \frac{\partial v_\vartheta}{\partial x} - \nu \frac{\partial^2 v_\vartheta}{\partial x^2} = \nu\left(\frac{\partial^2 v_\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\vartheta}{\partial r} - \frac{v_\vartheta}{r^2}\right) \quad (3.2)$$

where ν is the coefficient of kinematic viscosity.

Owing to the effect of viscosity during the motion of the flow along axis x, the nonuniformity of tangential velocities will be diminishing, $\partial v_\vartheta / \partial x \to 0$, and the profile of the tangential velocity v_ϑ approaches the law of the rotation of solids, $\lim_{x \to \infty} v_\vartheta = \omega r$, where ω is the angular velocity of rotation.

The process of balancing some given small nonuniformity of velocities along x when $v_r' r/\nu < 1$ (v_r' is the difference of the radial velocity from zero) is defined by the differential equation (3.2) the solution of which is carried out as shown in Section 1.

In the particular case when the profiles of the velocity $v_\varphi - \omega r$ vary along axis x in a like manner, the velocity profiles are defined by the following relation upon withdrawal from the source of the flow of rotation



$$v_\varphi = \omega r + c_0 \exp\left[\left(\frac{r_0 v_0}{2\nu} \pm \pm \sqrt{\frac{r_0^2 v_0^2}{4\nu^2} + 3.39}\right)\frac{x}{r_0}\right] J_1\left(1.84\,\frac{r}{r_0}\right) \qquad (3.3)$$

Figure 3.

Plus sign in the relation (3.3) corresponds to $x < 0$ and minus sign - to $x > 0$, i.e. the more intensive balancing of the nonuniformity of the tangential velocities occurs in the direction opposite the flow of the fluid.

It follows from the relation (3.3) that the rate of balancing increases with the diminution of the number for $R = r_0 v_0 / \nu$.

The balancing of the parameters of the rotational turbulent flow, in which unlike the laminar viscous flow [6] attenuation will be inversely proportional to x

$$(v_\varphi - \omega r) \sim \frac{1}{x}$$

can be examined in a similar manner.

Submitted 14 April 1959

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